

ON SEMI-BARRELLED SPACES

HENRI BOURLÈS

ABSTRACT. The aim of this paper is to clarify the properties of semi-barrelled spaces (also called countably quasi-barrelled spaces in the literature). These spaces were studied by several authors, in particular in the classical book of N. Bourbaki "Espaces vectoriels topologiques". However, six incorrect statements can be found in this reference. In particular: a Hausdorff and quasi-complete semi-barrelled space is complete, a semi-barrelled, semi-reflexive space is complete, a locally convex hull of semi-barrelled semi-reflexive spaces is semi-reflexive, a locally convex hull of semi-barrelled reflexive spaces is reflexive. We show through counterexamples that these statements are false. To conclude, we show how these false claims can be corrected and we collect some properties of semi-barrelled spaces.

1. INTRODUCTION

Semi-barrelled spaces are studied in the classical book of N. Bourbaki [1] ("Espaces vectoriels topologiques"), in §IV.3, n°1 ("Espaces semi-tonnelés"). A locally convex space E is said to be semi-barrelled if the following condition holds: (a) let U be a bornivorous part of E which is the intersection of a sequence of convex balanced closed neighborhoods of 0 in E ; then U is a neighborhood of 0 in E . Chronologically, (\mathcal{DF}) spaces were introduced before semi-barrelled spaces, by Grothendieck [3]: a locally convex space E is a (\mathcal{DF}) space if (a) holds and (b) the canonical bornology of E has a countable base. Then Husain [5] considered spaces satisfying (a) but not (b) and called them "countably semi-barrelled spaces". Thus, these spaces are called "semi-barrelled spaces" by Bourbaki [1], and we adopt the latter terminology in the sequel.

Bourbaki's account on semi-barrelled spaces consists of the above-quoted section, where three equivalent definitions are given, and of nine statements (Exerc. 9, p. IV.60). Six of them are not correct, therefore a clarification of the properties of these spaces is needed and presented below. These statements are the following ones:

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(1) Let E be a locally convex Hausdorff semi-barrelled space, M be a closed subspace of E , E' be the dual of E . The strong topology $\beta(M^0, E/M)$ is identical to the topology induced on M^0 by the strong topology $\beta(E', E)$.

(2) Let E be a locally convex Hausdorff space, M a (non necessarily closed) vector subspace of E . If M is a semi-barrelled space, then the strong topology $\beta(E'/M^0, M)$ is identical to the quotient topology by M^0 of the strong topology $\beta(E', E)$.

(3) A Hausdorff and quasi-complete semi-barrelled space M is complete.

(4) Let E be a semi-barrelled Hausdorff space and M be a closed subspace of E . Then E/M is a semi-barrelled space.

(5) Let (E_n) be a sequence of semi-barrelled spaces, E be a vector space, and for each n , let f_n be a linear mapping from E_n into E . Suppose that E is the union of the $f_n(E_n)$. Then E is semi-barrelled for the finest locally convex topology for which all the f_n are continuous.

(6) The completion of a semi-barrelled Hausdorff space is semi-barrelled.

(7) A semi-barrelled, semi-reflexive space M is complete.

(8) Let (E_n) and E be as in (5). If each E_n is semi-reflexive and if E is Hausdorff, then E is semi-reflexive.

(9) Let (E_n) and E be as in (5). If each E_n is reflexive and if E is Hausdorff, then E is reflexive.

Claims (4) and (5) were proved in ([5], Thm. 8 and Corol. 14) with the sequences (E_n) and (f_n) replaced by non-necessarily countable families (E_α) and (f_α) in Claim (5) –and one can notice that it is not necessary to assume in Claim (4) that M is closed. Claim (6) was proved in ([6], Chap. V, Prop. 3, p. 133). We show below through counterexamples that Claims (1)-(3) and (7)-(9) are false. So, Bourbaki's account on semi-barrelled spaces is very misleading for people studying topological vector spaces, and a clarification of the properties of these spaces is needed. We conclude by additional remarks where we show how these false claims can be corrected and we collect some properties of semi-barrelled spaces.

2. COUNTEREXAMPLES

2.1. Counterexample to Claim (1). An example is given in ([11], §27, 2., p. 370) of a Montel space F and a closed subspace H of F , such that the initial topology of F does not induce on H its strong topology. So, let E be the strong dual of F and $M = H^0$. Then E is a Montel space, thus it is barrelled, M is a closed subspace of E , $H = M^0$ according to the

bipolar theorem, and the strong topology $\beta(M^0, E/M)$ is not identical to the topology induced on M^0 by the strong topology $\beta(E', E)$.

2.2. Counterexample to Claim (2). It is well-known that there exist non-distinguished Fréchet spaces ([11], §31, 7., p. 435). Such a space M can be embedded in the product E of a family of Banach spaces ([11], §18, 3., (7), p. 208). Then the strong dual E'_β is a locally convex direct sum of Banach spaces, thus is barrelled; therefore E'_β/M^0 (the topology \mathfrak{T}_1 of which is the quotient topology of $\beta(E', E)$ by M^0) is barrelled. Since M is non-distinguished, its strong dual (the topology \mathfrak{T}_2 of which is $\beta(E'/M^0, M)$) is not barrelled. Therefore the topologies \mathfrak{T}_1 and \mathfrak{T}_2 are not identical.

2.3. Counterexample to Claims (3) and (7). An example of a non-complete Montel space has been given in ([10], §5). This space is barrelled and reflexive, hence quasi-complete ([1], p. IV.16, Remarque 2). Therefore, there exist reflexive (thus quasi-complete) barrelled spaces which are not complete.

2.4. Counterexample to Claims (8) and (9). If Claim (8) is correct, so is Claim (9) since E is barrelled if the spaces E_n are barrelled ([11], §27, 1., (3), p. 368). However, in ([11], §31, 5., p. 434), an example is given of a Fréchet-Montel space $E_1 = \lambda$ and a closed subspace N of E_1 such that $E = E_1/N$ is topologically isomorphic to l^1 , thus is not reflexive. The space E_1 is reflexive and $E = f_1(E_1)$, where f_1 is the canonical surjection, is not. Therefore, Claim (9) is false and Claim (8) is false too.

3. CONCLUDING REMARKS

According to the proof of ([3], corol., p. 79), Claims (8) and (9) are correct if E and the sequence (E_n) are as in (5), assuming that every bounded part of E is included in the closed balanced convex hull of a finite number of $f_n(B_n)$ where each B_n is a bounded part of E_n (in particular, this holds if E is a regular inductive limit of the E_n [2], and *a fortiori* if the E_n are closed subspaces of E and E is the strict inductive limit of the E_n). On the other hand, all statements are correct if (\mathcal{DF}) spaces are considered in place of semi-barrelled spaces ([3], prop. 5, p. 76 and corol., p. 79), ([4], corol. 2, p. 170).

A closed subspace F of a semi-barrelled space E is not necessarily semi-barrelled, as shown in ([3], p. 97) and ([7], (iii)), except if F is finite-codimensional ([15], Thm. 6, p. 169) or if F is countable-codimensional and such that for every bounded subset B of E , F is finite-codimensional

in the space spanned by $F \cup B$ ([14], Thm. 3). Every infra-barrelled (also called *quasi-barrelled*) space is semi-barrelled ([5], Prop. 2, p. 292), every infra-barrelled space is a Mackey space ([4], p. 107), but a semi-barrelled space is not necessarily a Mackey space by ([3], Remarque 8, p. 74) – since neither is a (\mathcal{DF}) space. Further results on semi-barrelled spaces can be found in, e.g., [13], [12], [8], and ([9], Chap. III, p. 33 and n°41., 44.; Chap. IV, 16.; Chap. VI, 4.).

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SATIE, ENS DE CACHAN/CNAM, 61 AVENUE PRÉSIDENT WILSON, F-94230
CACHAN, FRANCE

E-mail address: `henri.bourles@satie.ens-cachan.fr`